Profit Maximization of My Seed-Planting Activity Business

1 Introduction and Rationale

This summer, my friends and I were discussing how the weather had become increasingly hot and unpredictable. We recognized that global warming, caused by rising concentrations of atmospheric greenhouse gases, was damaging both our planet and our daily lives. We concluded that raising money for a tree-planting charity would contribute to reducing greenhouse gases and thereby help to mitigate global warming.

To raise awareness of the importance of climate change, we organized a seed-planting activity stall that sold small and large flowerpots and offered visitors a planting experience. The business aimed both to promote the importance of planting and to raise money for the tree-planting charity. When we started selling, we made a loss, but as we scaled up production, our profit began to increase. However, we could not simply keep increasing production to maximize profit, because we relied on a monthly allowance and therefore had a limited budget. This led me to investigate how the production of the two goods could be optimized to maximize profit within a limited budget. To answer this question, I constructed a mathematical model that determines the optimal production of each good under our current resource limits, and hence the maximum profit our business can achieve. With this model, we can maximize the profit available for donation and so raise as much money as possible.

2 Aim and Approach

The aim of this investigation is to determine the combination of the two products that maximizes weekly profit under a resource constraint, and to find the maximum weekly profit that can be achieved. To do so, I construct a mathematical model of the relationship between profit and the quantities produced. Using weekly financial data from the business, I estimate a total revenue function and a total cost function for both small and large flowerpots, and from these I derive a profit function. I then maximize the profit function

subject to the constraint by the method of Lagrange multipliers to find the production levels of small and large flowerpots at the critical point. Finally, I use the bordered Hessian test to verify that this critical point is a maximum.

3 Mathematical Background

3.1 Functions of Two Variables and Partial Derivatives

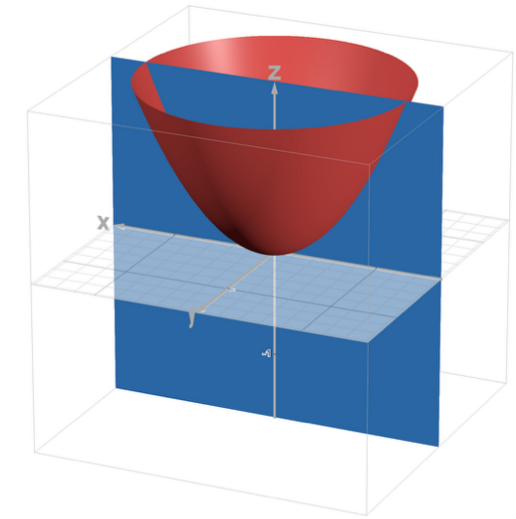


Figure 1: The function $f(x, y) = x^2 + y^2$ and the plane $y = 0$, plotted in Desmos.

A function of two variables takes two independently chosen inputs and maps them to a single output. For a pair of independent variables (x, y), the output can be written $z = f(x, y)$ and visualized in a three-dimensional Cartesian coordinate system in which the x- and y-axes form the base plane and the z-axis represents the height of the function (Figure 1). In this investigation there are two input variables: the quantity of small flowerpots produced, q_s, and the quantity of large flowerpots produced, q_ℓ.

The rate of change of such a function with respect to one variable, while the other is held fixed, is given by a partial derivative (University of Melbourne, n.d.). Imagine moving across the surface $z = f(x, y)$ in a direction parallel to the x-axis through the point (x_0, y_0). Along this path, y remains equal to y_0 while x changes, so y_0 is treated as a constant. The instantaneous rate of change of f in the x-direction is therefore the derivative of $f(x, y_0)$ with respect to x. This is the partial derivative of f with respect to x, denoted $\partial f/\partial x$ or f_x; the partial derivative with respect to y is denoted $\partial f/\partial y$ or f_y. Geometrically, as shown in Figure 2, the vertical plane $y = y_0$ intersects the surface in a curve AP_0B, and

$\frac{\partial f}{\partial x}(x_0, y_0)$ is the slope of the tangent P_0T to this curve at the point (x_0, y_0) (University of Melbourne, n.d.).

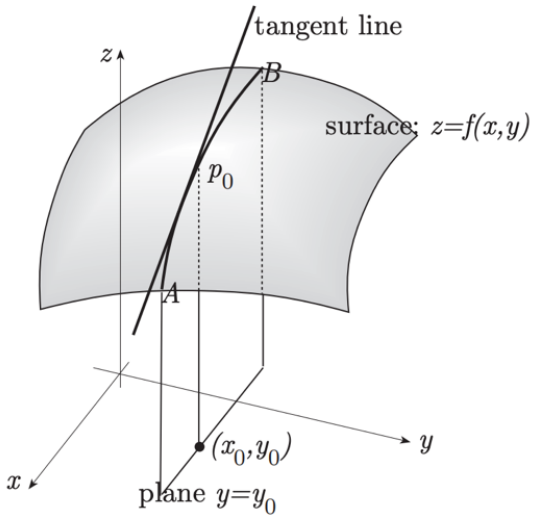


Figure 2: Tangent to a surface in the direction of the x-axis (University of Melbourne, n.d.).

To maximize such a function, we need the values of x and y that give the largest value of $f(x, y)$. A critical point is a point at which the rate of change of a function is zero or undefined (Vanberg, n.d.). For a differentiable function of two variables, critical points occur where both partial derivatives f_x and f_y are zero simultaneously, since the function then has zero rate of change in both the x- and y-directions. Such points are candidates for local maxima, local minima or saddle points (May & Bart, n.d.).

3.2 Constrained Optimization

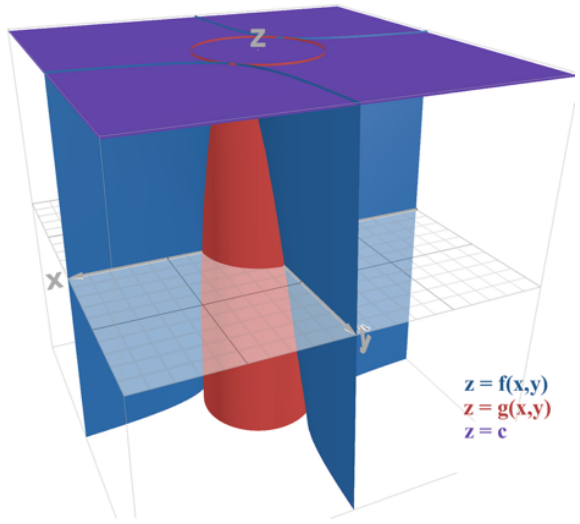


Figure 3: $z = f(x, y)$, $z = g(x, y)$ and $z = c$, plotted in Desmos.

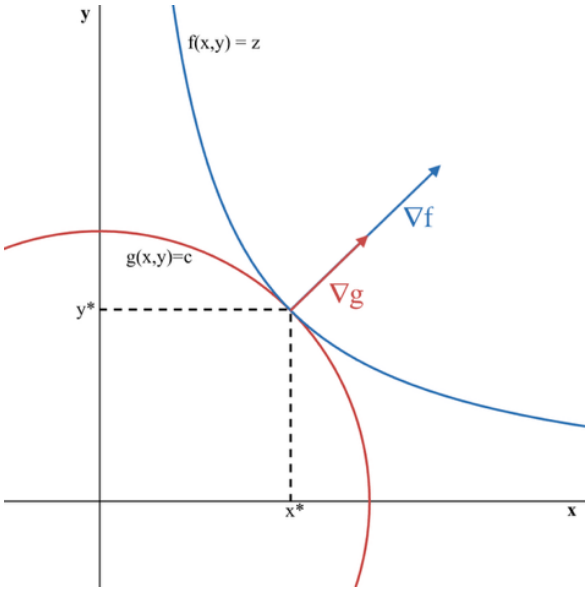


Figure 4: Level curves of f and g at the constrained maximum.

Constrained optimization involves optimizing an objective function subject to a constraint. In this investigation, I optimize the production quantities q_s and q_ℓ under the condition that the number of products that can be made is restricted by limited resources. To visualize this, consider an objective function $z = f(x, y)$ subject to a constraint $g(x, y) = c$ (Figure 3). Figure 4 shows level curves of f and g: a level curve of f is a set of points on which $f(x, y) = k$ for some constant k, and the constraint curve is the set of points on which $g(x, y) = c$. Figures 3 and 4 show that f attains its maximum subject to $g(x, y) = c$ at the point (x^*, y^*), where increasing f is balanced against remaining on the constraint curve. Geometrically, at this point the level curve of f just touches the constraint curve without crossing it. Since the gradient of a function is perpendicular to its level curves, the gradients ∇f and ∇g are parallel at (x^*, y^*), so one is a scalar multiple of the other:

$$\nabla f = \lambda \nabla g, \tag{1}$$

where the scalar λ is called the Lagrange multiplier. The Lagrange multiplier measures the rate at which the maximum value of f increases as the constraint constant c is increased slightly. The condition $\nabla f = \lambda \nabla g$ identifies the point of tangency, at which moving away from (x^*, y^*) along the constraint cannot increase the objective function.

In component form, at the maximum each partial derivative of f is λ times the corre-

sponding partial derivative of g:

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}. \tag{2}$$

Rearranging,

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0, \qquad \text{that is,} \quad \nabla f - \lambda \nabla g = \mathbf{0}. \tag{3}$$

Now consider the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda\Big(g(x, y) - c\Big), \tag{4}$$

where $f(x, y)$ is the objective function (here, profit) and $g(x, y) - c = 0$ is the constraint (here, resource capacity). The Lagrangian combines the objective function and the constraint into a single function. Setting all of its partial derivatives equal to zero gives

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0, \qquad \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \lambda} = -\Big(g(x, y) - c\Big) = 0. \tag{5}$$

The first two equations are exactly the condition $\nabla f - \lambda \nabla g = \mathbf{0}$, and the third recovers the constraint $g(x, y) = c$. These are therefore the conditions required to maximize $f(x, y)$ subject to $g(x, y) = c$, and solving them gives the critical points of the constrained problem.

The nature of a critical point (x^*, y^*) can then be determined using the bordered Hessian test. The Hessian matrix is the square matrix of second-order partial derivatives of a scalar-valued function, and it describes the local curvature of a multivariable function (Douglas, 2023). In unconstrained optimization, the Hessian is used to classify a critical point as a maximum, a minimum or a saddle point. The bordered Hessian extends this idea to problems with equality constraints: it combines the second-order partial derivatives of the Lagrangian with the first-order partial derivatives of the constraint function, so that the second-order conditions take the constraint into account (Cooke, n.d.). For a problem with two variables and one constraint, the sign of the determinant of the bordered Hessian classifies the critical point within the feasible set defined by the constraint.

4 Financial Data of the Business

Tables 1 and 2 give the total cost and total revenue for small and large flowerpots respectively over the 11 weeks of selling. Table 1 gives the quantity of small flowerpots produced (q_s) and sold, and Table 2 gives the quantity of large flowerpots produced (q_ℓ) and sold. Where rounding is required for practical purposes, monetary values are rounded

to 2 decimal places, the standard for Hong Kong dollars (HKD); intermediate values are carried to 6 decimal places for accuracy.

Week	1	2	3	4	5	6	7	8	9	10	11
Produced (q_s)	16	32	36	57	70	65	45	11	22	52	60
Sold	16	32	36	56	69	65	43	11	22	52	59
Cost	488.04	651.08	691.84	905.83	1038.30	987.35	783.55	437.09	549.18	854.88	936.40
Revenue	480	960	1080	1680	2070	1950	1290	330	660	1560	1770

Table 1: Weekly production, sales, total cost and total revenue (HKD) for small flowerpots.

Week	1	2	3	4	5	6	7	8	9	10	11
Produced (q_ℓ)	6	15	18	23	12	17	28	37	34	26	22
Sold	6	15	17	23	12	17	28	37	34	25	22
Cost	459.28	660.70	727.84	839.74	593.56	705.46	951.64	1153.06	1085.92	906.88	817.36
Revenue	300	750	850	1150	600	850	1400	1850	1700	1250	1100

Table 2: Weekly production, sales, total cost and total revenue (HKD) for large flowerpots.

5 Modelling the Total Revenue Function

A total revenue function is needed to model profit, and it can be estimated from the business data. The relationship between q_s and revenue is not perfectly linear, because revenue depends on the quantity sold rather than the quantity produced. Nevertheless, since almost every product made was sold, revenue is very close to a linear function of q_s, so it is reasonable to model it in the form $R_s(q_s) = p_s q_s + k_s$, where p_s is approximately the price. I performed the linear regression on a TI-Nspire CX II graphing calculator.

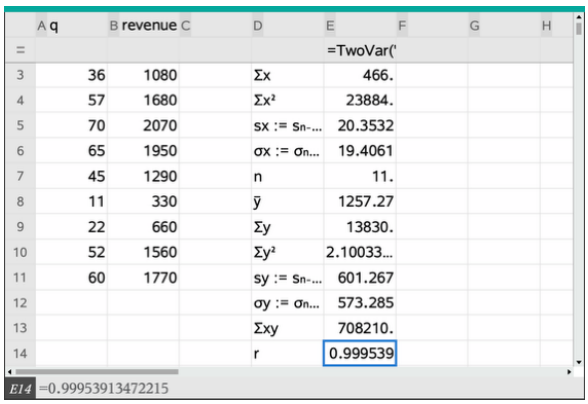


Figure 5: Regression output for revenue against q_s.

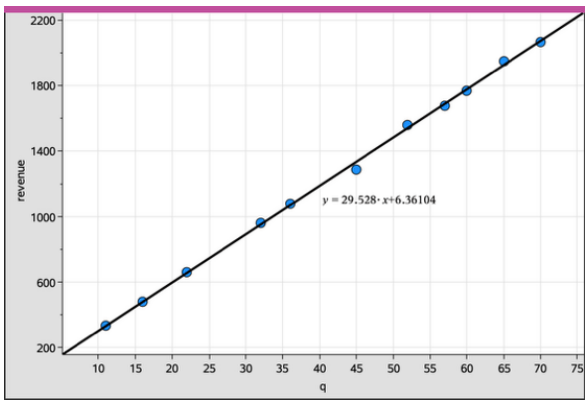
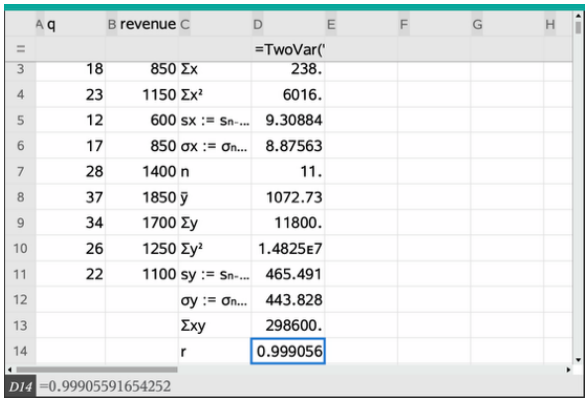


Figure 6: Revenue against q_s, with fitted line.

There is a very strong positive correlation between q_s and the revenue R_s from small flowerpots ($r \approx 0.999539$), and $r^2 \approx 0.999078$, so the linear model explains about 99.91% of the variance in revenue. A linear model is therefore appropriate, and polynomial regression would add complexity without any practical benefit. The revenue function for small flowerpots is

$$R_s(q_s) \approx 29.528q_s + 6.36104. \tag{6}$$



	A q	B revenue	C	D	E	F	G	H
=				=TwoVar('				
3	18	850	Σx	238.				
4	23	1150	Σx²	6016.				
5	12	600	sx := sn-...	9.30884				
6	17	850	σx := σn...	8.87563				
7	28	1400	n	11.				
8	37	1850	ȳ	1072.73				
9	34	1700	Σy	11800.				
10	26	1250	Σy²	1.4825E7				
11	22	1100	sy := sn-...	465.491				
12			σy := σn...	443.828				
13			Σxy	298600.				
14			r	0.999056				

D14 =0.99905591654252

Figure 7: Regression output for revenue against q_ℓ.

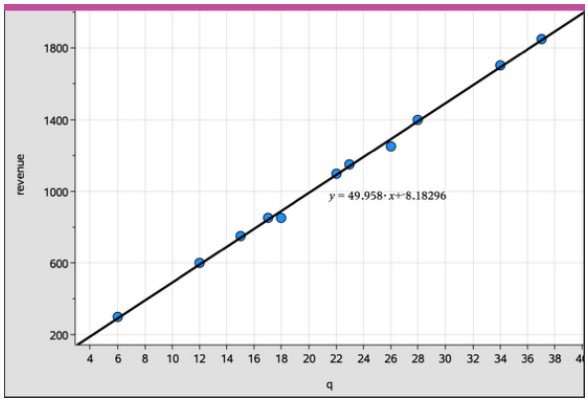


Figure 8: Revenue against q_ℓ, with fitted line.

I performed the same linear regression for q_ℓ and the revenue R_ℓ from large flowerpots. There is again a very strong positive correlation ($r \approx 0.999056$, $r^2 \approx 0.998113$), so the model explains about 99.81% of the variance in revenue. The calculator displays the fitted line as $y = 49.958x + {}^{-}8.18296$, where $^{-}$ is the TI-Nspire negative sign, so the intercept is negative. The revenue function for large flowerpots is

$$R_\ell(q_\ell) \approx 49.958q_\ell - 8.18296. \tag{7}$$

6 Modelling the Total Cost Function

To model total cost, I needed a cost function for each type of flowerpot. The cost of producing small flowerpots, C_s, showed a perfect linear relationship with q_s, because the variable cost is directly proportional to the quantity produced. I therefore performed linear regression on the same calculator and modelled each cost function in the form

$$C(q) = (\text{unit cost}) \times q + \text{fixed cost}. \tag{8}$$

	A q	B cost	C	D	E	F	G	H
=				=TwoVar('				
3	36	691.84	Σx	466.				
4	57	905.83	Σx²	23884.				
5	70	1038.3	sx := sn-...	20.3532				
6	65	987.35	σx := σn...	19.4061				
7	45	783.55	n	11.				
8	11	437.09	ȳ	756.685				
9	22	549.18	Σy	8323.54				
10	52	854.88	Σy²	6.72845...				
11	60	936.4	sy := sn-...	207.4				
12			σy := σn...	197.748				
13			Σxy	394828.				
14			r	1.				

D14 =1.

Figure 9: Regression output for cost against q_s.

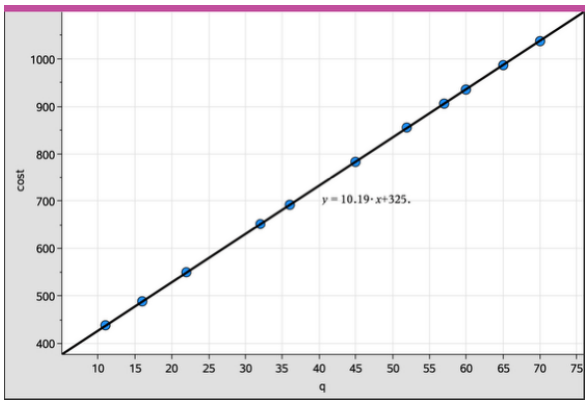


Figure 10: Cost against q_s, with fitted line.

The correlation between q_s and C_s is perfect ($r = 1$, so $r^2 = 1$), so the cost model fits the data exactly. The cost function for small flowerpots is

$$C_s(q_s) = 10.19q_s + 325. \tag{9}$$

	B cost	C	D	E	F	G	H	I
=				=TwoVar('				
3	727.84		Σx	238.				
4	839.74		Σx²	6016.				
5	593.56		sx := sn-...	9.30884				
6	705.46		σx := σn...	8.87563				
7	951.64		n	11.				
8	1153.06		ȳ	809.222				
9	1085.92		Σy	8901.44				
10	906.88		Σy²	7.63726...				
11	817.36		sy := sn-...	208.332				
12			σy := σn...	198.637				
13			Σxy	211988.				
14			r	1.				

E14 =1.

Figure 11: Regression output for cost against q_ℓ.

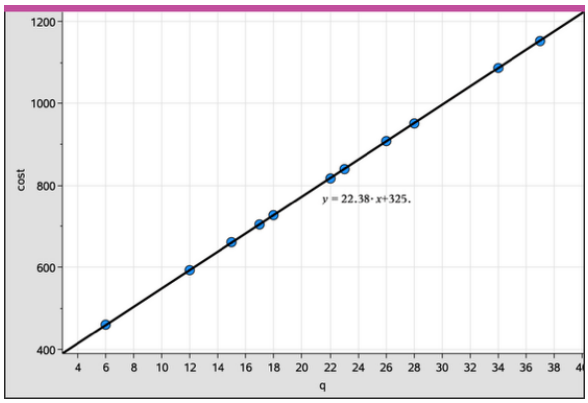


Figure 12: Cost against q_ℓ, with fitted line.

The same regression for large flowerpots also gives $r^2 = 1$. The cost function for large flowerpots is

$$C_\ell(q_\ell) = 22.38q_\ell + 325. \tag{10}$$

The vertical intercept of 325 in each cost function is the share of the weekly fixed cost allocated to that product. Together they give a total fixed cost of $325 + 325 = 650$ HKD per week, consisting of 150 HKD spent on soil and 500 HKD paid in rent, which remained constant regardless of the level of production.

7 Modelling the Profit Function

Let $\pi(q_s, q_\ell)$ denote the weekly profit from producing q_s small and q_ℓ large flowerpots. Profit is total revenue minus total cost:

$$\pi(q_s, q_\ell) = \Big(R_s(q_s) + R_\ell(q_\ell)\Big) - \Big(C_s(q_s) + C_\ell(q_\ell)\Big). \tag{11}$$

Substituting the fitted functions,

$$\begin{aligned}\pi(q_s, q_\ell) &= \Big(29.528q_s + 6.36104 + 49.958q_\ell - 8.18296\Big) - \Big(10.19q_s + 325 + 22.38q_\ell + 325\Big) \\ &= 19.338q_s + 27.578q_\ell - 651.82192. \end{aligned} \tag{12}$$

8 Deriving the Resource Constraint

Production of small and large flowerpots was constrained by limited resources: in every week after Week 4, we ran out of soil and seeds. To model this constraint realistically, I needed an equation that captures the non-linear trade-off in resource use between small and large flowerpots. Resources used for one product cannot always be transferred to the other (for example, a small pot cannot be used to make a large flowerpot). A large shift in resources towards one product therefore causes a larger fall in total production because of inefficiency: the unused resources for the other product still count towards cost but no longer contribute to output. This is reflected in Table 3, where total production varies from week to week while total resource usage stays at around 5000 units once resources became limiting.

To model this non-linear trade-off, I used a constraint of the form $aq_s^2 + bq_\ell^2 = r$, where r is the total resource usage. In this model, a small increase in the production of one good requires an increasingly large fall in the production of the other, and vice versa, so the quadratic form captures the combined, non-linear effect of both products on resources. Since each large flowerpot uses twice the resources of a small flowerpot (twice the soil, seeds and string),

$$\text{resources per small flowerpot} \propto 1, \qquad \text{resources per large flowerpot} \propto 2. \tag{13}$$

To capture the quadratic trade-off, I squared the resource usage of each product:

$$\begin{aligned}\text{total resource usage for small flowerpots} &\propto q_s^2, \\ \text{total resource usage for large flowerpots} &\propto (2q_\ell)^2. \end{aligned} \tag{14}$$

Total resource usage is therefore modelled as $q_s^2 + (2q_\ell)^2 = r$. This constraint does not

reflect the trade-off perfectly, because it assumes that the trade-off between q_s and q_ℓ follows the same fixed form at every level of production. It is nevertheless useful for estimating the production levels that maximize profit.

Week	1	2	3	4	5	6	7	8	9	10	11
q_s	16	32	36	57	70	65	45	11	22	52	60
q_ℓ	6	15	18	23	12	17	28	37	34	26	22
$q_s + q_\ell$	22	47	54	80	82	82	73	48	56	78	82
$r = q_s^2 + (2q_\ell)^2$	400	1924	2592	5365	5476	5381	5161	5597	5108	5408	5536

Table 3: Weekly production and total resource usage.

Table 3 shows that the total resource usage reached a maximum of 5597 units (in Week 8), so the business is capable of production with a resource usage of 5597 units. The resource constraint is therefore estimated as

$$q_s^2 + (2q_\ell)^2 = 5597. \tag{15}$$

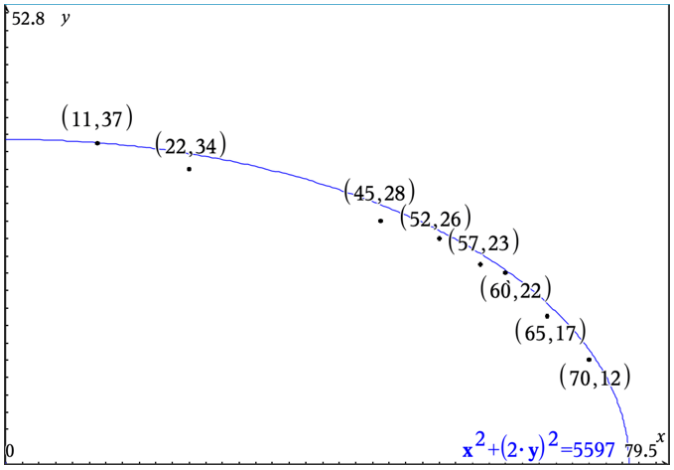


Figure 13: The constraint curve $g(q_s, q_\ell) = 5597$ compared with the observed values of (q_s, q_ℓ).

Writing the constraint in the form $g(x, y) = c$,

$$g(q_s, q_\ell) = q_s^2 + (2q_\ell)^2 = 5597, \qquad \text{that is,} \qquad g(q_s, q_\ell) - 5597 = 0. \tag{16}$$

Figure 13 plots the observed values of q_s and q_ℓ together with the constraint curve. Although the constraint is not a perfectly accurate model, it describes the data reasonably well, so it can be used to approximate the production levels that maximize profit.

9 First-Order Conditions and Optimal Production

To maximize $\pi(q_s, q_\ell)$ subject to $g(q_s, q_\ell) = 5597$, I used the Lagrangian function

$$\mathcal{L}(q_s, q_\ell, \lambda) = \pi(q_s, q_\ell) - \lambda\Big(g(q_s, q_\ell) - 5597\Big). \tag{17}$$

Substituting the profit function and the constraint,

$$\begin{aligned}\mathcal{L}(q_s, q_\ell, \lambda) &= \Big(19.338q_s + 27.578q_\ell - 651.82192\Big) - \lambda\Big(q_s^2 + (2q_\ell)^2 - 5597\Big) \\ &= 19.338q_s + 27.578q_\ell - 651.82192 - \lambda q_s^2 - 4\lambda q_\ell^2 + 5597\lambda. \end{aligned} \tag{18}$$

To find the critical points, I differentiated the Lagrangian partially with respect to q_s, q_ℓ and λ and set each derivative equal to zero:

$$\frac{\partial \mathcal{L}}{\partial q_s} = 19.338 - 2\lambda q_s = 0 \qquad \Longrightarrow \quad q_s = \frac{19.338}{2\lambda}, \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial q_\ell} = 27.578 - 8\lambda q_\ell = 0 \qquad \Longrightarrow \quad q_\ell = \frac{27.578}{8\lambda}, \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -q_s^2 - 4q_\ell^2 + 5597 = 0 \qquad \Longrightarrow \quad q_s^2 + 4q_\ell^2 = 5597. \tag{21}$$

The last condition is the constraint $g(q_s, q_\ell) - 5597 = 0$. Substituting the expressions for q_s and q_ℓ into the constraint gives an equation for the Lagrange multiplier:

$$\begin{aligned} q_s^2 + 4q_\ell^2 &= \left(\frac{19.338}{2\lambda}\right)^2 + 4\left(\frac{27.578}{8\lambda}\right)^2 = \frac{(19.338)^2}{4\lambda^2} + \frac{4(27.578)^2}{64\lambda^2} \\ &\approx \frac{373.958244}{4\lambda^2} + \frac{760.546084}{16\lambda^2} = \frac{4 \times 373.958244 + 760.546084}{16\lambda^2} \approx \frac{2256.379060}{16\lambda^2}. \end{aligned} \tag{22}$$

Hence

$$\frac{2256.379060}{16\lambda^2} = 5597 \quad \Longrightarrow \quad \lambda^2 = \frac{2256.379060}{16 \times 5597} \approx 0.025196, \tag{23}$$

so that

$$\lambda \approx \pm 0.158733. \tag{24}$$

Only the positive value is meaningful. If $\lambda < 0$, then $q_s = 19.338/(2\lambda)$ and $q_\ell = 27.578/(8\lambda)$ would both be negative, which is impossible for production quantities (this solution corresponds to the minimum of profit on the constraint curve). Economically, a positive λ also means that increasing the available resources increases the maximum profit, as expected. Therefore

$$\lambda \approx 0.158733. \tag{25}$$

Substituting λ into the first-order conditions gives the profit-maximizing production levels:

$$q_s = \frac{19.338}{2\lambda} \approx 60.913447, \qquad q_\ell = \frac{27.578}{8\lambda} \approx 21.717228. \tag{26}$$

Substituting these values into the profit function (12) gives the maximum profit:

$$\begin{aligned} \pi(60.913447,\, 21.717228) &= 19.338 \times 60.913447 + 27.578 \times 21.717228 - 651.82192 \\ &\approx 1125.04 \text{ HKD}. \end{aligned} \tag{27}$$

The value of λ also has a useful interpretation: each additional unit of resource capacity would increase the maximum weekly profit by approximately 0.16 HKD.

10 Second-Order Conditions

To verify the nature of the critical point (60.913447, 21.717228), I applied the bordered Hessian test. First, the second-order partial derivatives of the Lagrangian are

$$\frac{\partial^2 \mathcal{L}}{\partial q_s^2} = -2\lambda, \qquad \frac{\partial^2 \mathcal{L}}{\partial q_\ell^2} = -8\lambda, \qquad \frac{\partial^2 \mathcal{L}}{\partial q_s\, \partial q_\ell} = \frac{\partial^2 \mathcal{L}}{\partial q_\ell\, \partial q_s} = 0. \tag{28}$$

The first-order partial derivatives of the constraint function $g(q_s, q_\ell) = q_s^2 + 4q_\ell^2$ are

$$\frac{\partial g}{\partial q_s} = 2q_s, \qquad \frac{\partial g}{\partial q_\ell} = 8q_\ell. \tag{29}$$

The bordered Hessian matrix is

$$\bar{H} = \begin{pmatrix} 0 & \dfrac{\partial g}{\partial q_s} & \dfrac{\partial g}{\partial q_\ell} \\[2ex] \dfrac{\partial g}{\partial q_s} & \dfrac{\partial^2 \mathcal{L}}{\partial q_s^2} & \dfrac{\partial^2 \mathcal{L}}{\partial q_s\, \partial q_\ell} \\[2ex] \dfrac{\partial g}{\partial q_\ell} & \dfrac{\partial^2 \mathcal{L}}{\partial q_\ell\, \partial q_s} & \dfrac{\partial^2 \mathcal{L}}{\partial q_\ell^2} \end{pmatrix} = \begin{pmatrix} 0 & 2q_s & 8q_\ell \\ 2q_s & -2\lambda & 0 \\ 8q_\ell & 0 & -8\lambda \end{pmatrix}. \tag{30}$$

For a problem with two variables and one constraint, a critical point is a local maximum if $\det \bar{H} > 0$ and a local minimum if $\det \bar{H} < 0$. Expanding along the first row,

$$\begin{aligned}\det \bar{H} &= 0 \cdot \left(\frac{\partial^2 \mathcal{L}}{\partial q_s^2} \frac{\partial^2 \mathcal{L}}{\partial q_\ell^2} - \left(\frac{\partial^2 \mathcal{L}}{\partial q_s \, \partial q_\ell} \right)^2 \right) - \frac{\partial g}{\partial q_s} \left(\frac{\partial g}{\partial q_s} \frac{\partial^2 \mathcal{L}}{\partial q_\ell^2} - \frac{\partial^2 \mathcal{L}}{\partial q_s \, \partial q_\ell} \frac{\partial g}{\partial q_\ell} \right) \\ &\quad + \frac{\partial g}{\partial q_\ell} \left(\frac{\partial g}{\partial q_s} \frac{\partial^2 \mathcal{L}}{\partial q_\ell \, \partial q_s} - \frac{\partial^2 \mathcal{L}}{\partial q_s^2} \frac{\partial g}{\partial q_\ell} \right) \\ &= - \left(\frac{\partial g}{\partial q_s} \right)^2 \frac{\partial^2 \mathcal{L}}{\partial q_\ell^2} + 2 \, \frac{\partial g}{\partial q_s} \frac{\partial g}{\partial q_\ell} \frac{\partial^2 \mathcal{L}}{\partial q_s \, \partial q_\ell} - \left(\frac{\partial g}{\partial q_\ell} \right)^2 \frac{\partial^2 \mathcal{L}}{\partial q_s^2}. \end{aligned} \tag{31}$$

Substituting the derivatives above,

$$\det \bar{H} = -(2q_s)^2(-8\lambda) + 0 - (8q_\ell)^2(-2\lambda) = 32\lambda q_s^2 + 128\lambda q_\ell^2. \tag{32}$$

At the critical point, $2q_s \approx 121.826894$, $8q_\ell \approx 173.737826$, $-2\lambda \approx -0.317467$ and $-8\lambda \approx -1.269867$, so

$$\begin{aligned}\det \bar{H} &\approx (121.826894)^2(1.269867) + (173.737826)^2(0.317467) \\ &\approx 18847.1 + 9582.7 \approx 28429.8 > 0. \end{aligned} \tag{33}$$

Since $\lambda > 0$, $\det \bar{H}$ is positive for any production levels that are not both zero, and in particular at the critical point. The critical point is therefore a maximum of profit subject to the resource constraint.

11 Profit Optimization

The model shows that, in theory, the business can achieve a maximum weekly profit of approximately 1125.04 HKD by producing about 60.91 small and 21.72 large flowerpots. In practice, however, flowerpots can only be produced in whole numbers, so the optimum must be an integer combination that does not exceed the available resources, that is, one satisfying $q_s^2 + 4q_\ell^2 \le 5597$. Rounding the continuous solution suggests the candidates in Table 4.

q_s	q_ℓ	$q_s^2 + 4q_\ell^2$	Profit (HKD)
61	22	5657	infeasible
61	21	5485	1106.93
60	22	5536	1115.17
59	23	5597	1123.41

Table 4: Integer production plans near the continuous optimum.

The combination $(61, 22)$ exceeds the resource limit, so it is not feasible. Of the feasible combinations near the continuous optimum, $(59, 23)$ gives the highest profit, 1123.41 HKD, and it uses exactly the available resources. Checking every feasible integer combination confirms that no other combination gives a higher profit under the model. The optimal weekly production plan is therefore 59 small flowerpots and 23 large flowerpots, giving a predicted maximum weekly profit of 1123.41 HKD, only 1.63 HKD below the theoretical (continuous) maximum. For comparison, the highest profit actually achieved in any week was 1116.24 HKD (Week 11, with 60 small and 22 large flowerpots produced).

12 Conclusion

In this investigation, I set out to find the combination of small and large flowerpots that maximizes the weekly profit of our seed-planting stall under a limited supply of resources. Using 11 weeks of financial data, I modelled total revenue and total cost with linear regression, both of which fitted the data almost perfectly ($r^2 > 0.998$). Combining them gave the profit function $\pi(q_s, q_\ell) = 19.338q_s + 27.578q_\ell - 651.82192$. The coefficients show that each large flowerpot contributes about 8.24 HKD more profit than a small one, but it also uses twice the resources, so profit cannot be maximized simply by producing as many large flowerpots as possible. I modelled this trade-off with the resource constraint $q_s^2 + (2q_\ell)^2 = 5597$, based on the maximum resource usage observed.

Using the method of Lagrange multipliers, I found the critical point $(q_s, q_\ell) \approx (60.91,\ 21.72)$ with $\lambda \approx 0.159$, and the bordered Hessian test ($\det \bar{H} \approx 28430 > 0$) confirmed that it is a maximum. The theoretical maximum weekly profit is about 1125.04 HKD. Since only whole flowerpots can be made, I compared the feasible integer combinations and found that producing **59 small and 23 large flowerpots** gives the highest profit, about 1123.41 HKD per week. This is slightly higher than the best profit we actually achieved (1116.24 HKD in Week 11), which suggests that our production in the later weeks was already close to optimal and that a small shift towards large flowerpots could raise a little more money for the tree-planting charity each week. The Lagrange multiplier adds a practical insight: each extra unit of resource capacity would add only about 0.16 HKD of profit, so increasing our budget for materials would help, but only gradually.

Through this investigation, I learned how multivariable calculus can be applied to a real decision that my friends and I faced. Partial derivatives, which first seemed abstract, gave a clear way to balance two products against a shared constraint, and the Lagrange multiplier turned out to have a meaningful economic interpretation rather than being only a tool for solving equations. I also saw the gap between a mathematical optimum and a practical one: the model's answer had to be adjusted to whole numbers and depends on

simplifying assumptions, especially the form of the resource constraint, which I discuss below.

13 Evaluation

The resource constraint assumes that the trade-off between q_s and q_ℓ follows the same fixed quadratic form at every level of production. This assumption is a theoretical simplification, and the constraint was estimated from only 11 weeks of data. In future, I could collect data over a longer period in order to model the constraint more accurately.

The continuous solution gives non-integer production levels, whereas in reality only whole flowerpots can be produced. I addressed this by comparing the feasible integer combinations near the continuous optimum, but a more rigorous approach would treat the problem as an integer optimization problem from the start.

The constraint may also overestimate the available resources: most of the observed data points lie inside the constraint curve rather than on it (Figure 13). A coefficient smaller than 2 on q_ℓ might represent the data more accurately. In future, I could test coefficients between 1 and 2 and select the curve that fits the observed data best.

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